Published in *INFORMS Transactions on Education* 18(2):127-141 (2018).
doi:10.1287/ited.2017.0182

Solving Smartphone Puzzle Apps by Mathematical Programming

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*supported by the HSBA Foundation

Abstract. This paper considers six logic puzzles (i.e., single player games) that are available as smartphone apps, namely *Thermometer Puzzles*, *Kakuro (Cross Sums)*, *Match 22: Color Puzzle Game*, ∞ *Infinity Loop*, *Slider*, and *Flow Free*. We provide mathematical models which can be applied to obtain solutions for these puzzles. In OR/MS lectures, the apps and models can be used as examples or exercises when teaching mathematical programming and for discussion of familiar model types such as shortest path and network flow models. Given the popularity of logic puzzle apps on smartphones, such exercises might be motivating for students. The level of difficulty of the models presented here varies from easy to more difficult.

Keywords. Puzzle, teaching modeling.

1 Introduction

Many logic puzzles can be modeled as integer programs, and solvers can be applied to find solutions. This has motivated OR/MS lecturers (including the author of this paper) to use such puzzles as interesting exercises in class. Dozens of logic puzzles along with IP models have been described in the literature, including the following: Chlond (2005) applies mathematical modelling techniques to Sudoku. The classical n queens problem is studied by Letavec and Ruggiero (2002), while Chlond and Toase (2002) look at chessboard placement puzzles in a more general context. DePuy and Taylor (2007) address board puzzles such as the Cracker Barrel peg game and the rush hour problem. The Fillomino puzzle is discussed by Pearce and Forbes (2017). Yeomans (2003) tackle what is usually referred to as "Einstein's riddle," while other logic grid puzzles are considered by Chlond (2014).

Most puzzles are originally paper-and-pencil or board games, but many are also available for smartphones, including the puzzles mentioned above. There are also some smartphone puzzles that have not been available as paper-and-pencil or board games. Logic puzzle apps are very popular—according to Google Play store, the ∞ *Infinity Loop* and *Flow Free* apps (which will be discussed in this paper) were installed more than 10,000,000 and 100,000,000 times, respectively. It might be especially interesting for students to work on OR exercises that are based on popular apps they can download on their phones or tablets.

In this contribution, we present six smartphone apps that can be used as entertaining and motivating examples and exercises in mathematical programming. The apps were selected with regard to several criteria. First, we wanted to achieve different levels of difficulty of the resulting

models, varying between easy and more difficult. Second, we wished to cover a broad range of problem types and modeling techniques. Thus we included different types of variables, mostly binary (as would probably be expected for logic games) but also integer and continuous. Also, we included specific problem types such as a shortest path and a network flow problem to allow for discussions of such familiar concepts in a rather unusual context. Third, we wanted to encourage students to work with the apps. Therefore we preferred puzzles with rules that are simple and easy to learn, and we chose apps that are available for free (all apps discussed here are available for Android for free, and some of them are available for iOS as well).

We proceed as follows: Section 2 presents the *Thermometer Puzzles* app which leads to a rather straighforward mathematical model. Then Section 3 describes a model for the *Kakuro (Cross Sums)* app; it could be used as an easy to moderate exercise in an introductory lecture. In Section 4, we discuss the app *Match 22: Color Puzzle Game* for which the model is a bit more advanced. This section also has a look at the ∞ *Infinity Loop* app to which a similar model can be applied. The last two apps are closely related to network models. Section 5 introduces the app *Slider* which can be modeled as a shortest path problem. This would be an easy exercise. Finally, Section 6 tackles the *Flow Free* app which leads to an extended network flow model that can be viewed as more difficult. It also allows to revisit the subtour elimination constraints known from the traveling salesman problem and to discuss the computational impact of different modeling approaches. The paper closes with some conclusions in Section 7.

2 Thermometer Puzzles

In the *Thermometer Puzzles* app, a grid with $n \times n$ cells is filled with thermometers. Each thermometer occupies two or more consecutive cells, either in a row or column (see Figure 1). The player has to determine which cells are filled with mercury. For each row and each column, the number of filled cells is given. Moreover, one has to take into account that thermometers are always filled from bottom to top (the bottom is indicated by a circle). That implies that if a cell is filled with mercury, then all cells towards the bottom of the same thermometer must be filled as well. Likewise, if a cell is not filled, then all cells towards the top must not be filled either.

In order to develop a mathematical model for this puzzle, we have to introduce some notation. Let R_i denote the filled cells in row i, and let C_j denote the filled cells in column j. Moreover, we let D be the set that includes all cell dependencies related to the thermometers. Each such dependency is given by a quadruple $(i, j, k, l) \in D$, reflecting that cell (k, l) can only be filled if cell (i, j) is filled as well.

The binary decision variables are defined by

$$x_{ij} = \begin{cases} 1 & \text{if cell } (i,j) \text{ is filled} \\ 0 & \text{otherwise} \end{cases}$$

As is often the case with puzzles, we are just looking for a feasible solution, so there is no optimization and thus no objective (that is, the objective can simply be 0). The constraints are as follows:

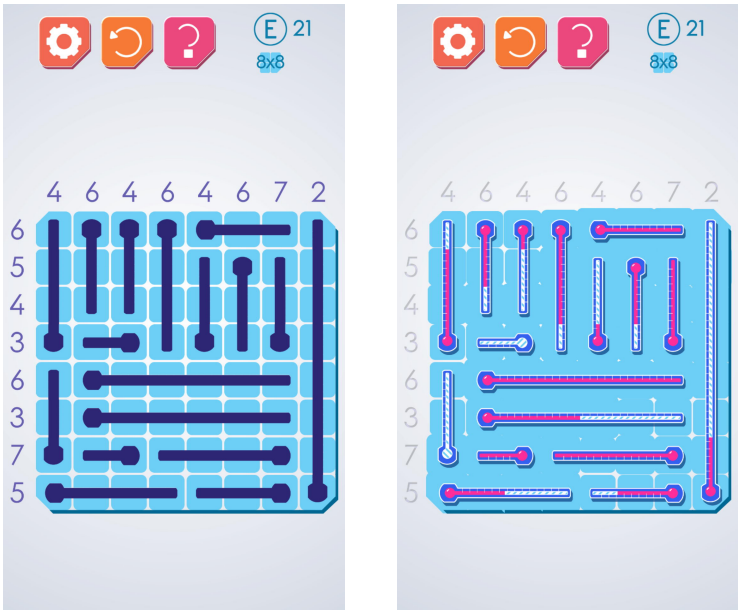
$$\sum_{j=1}^{n} x_{ij} = R_i \qquad i = 1, \ldots, n \tag{1}$$

$$\sum_{i=1}^{n} x_{ij} = C_j \qquad j = 1, \ldots, n \tag{2}$$

$$x_{ij} \geq x_{kl} \qquad (i, j, k, l) \in D \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad i, j = 1, \ldots, n \tag{4}$$

Constraints (1) and (2) take care of the required number of filled cells in each row and column, respectively. Then constraints (3) make sure that a cell (k, l) cannot be filled if cell (i, j) below



(a) Puzzle **(b)** Solution

Figure 1: Smartphone app *Thermometer Puzzles*

(k, l) in the same thermometer is not filled. Finally, constraints (4) define the decision variables to be binary.

3 Kakuro (Cross Sums)

Kakuro, also called Cross Sums, is a classical paper-and-pencil puzzle that has been around for decades. Several apps are available as well. We consider the app *Kakuro (Cross Sums)* displayed in Figure 2. The rules are as follows. We have an $n \times n$ grid in which the empty cells must be filled with digits (between 1 and 9). For each set of consecutive empty cells, either horizontal or vertical, a value is given. The sum of the digits in these cells must be equal to the given value. Moreover, the digits involved in such a sum must be different.

We use the following notation. Each empty cell is referred to as (i, j) where i is the row and j is the column in the grid. The set of all empty cells is denoted as C. Moreover, K is the number of sums in the grid (we do not have to distinguish between horizontal and vertical sums). For sum $k \in \{1, \ldots, K\}$, we have the related set of cells R_k and the resulting value of the sum r_k. Consider the first horizontal sum in the upper left corner of Figure 2 as an example. We have $k = 1$, $R_1 = \{(1,1), (1,2), (1,3), (1,4)\}$ and $r_1 = 11$. For notational convenience, we refer to the set of digits as D, thus $D = \{1, \ldots, 9\}$.

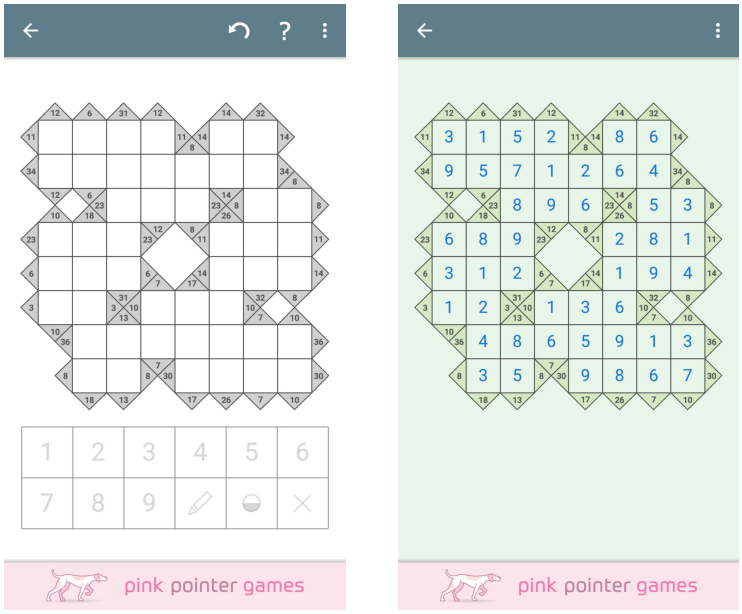
Next, we define the decision variables. We have a binary decsion variable for each cell $(i, j) \in C$ and each digit $d \in D$:

$$x_{ijd} = \begin{cases} 1 & \text{if digit } d \text{ is assigned to cell } (i,j) \\ 0 & \text{otherwise} \end{cases}$$

We obtain the following mathematical formulation (again, there is no objective function):

$$\sum_{d \in D} x_{ijd} = 1 \qquad (i,j) \in C \tag{5}$$

$$\sum_{(i,j) \in R_k} x_{ijd} \leq 1 \qquad k = 1, \ldots, K;\ d \in D \tag{6}$$



(a) Puzzle **(b)** Solution

Figure 2: Smartphone app *Kakuro (Cross Sums)*

$$\sum_{(i,j)\in R_k} \sum_{d\in D} d x_{ijd} = r_k \qquad k = 1,\ldots,K \tag{7}$$

$$x_{ijd} \in \{0,1\} \qquad (i,j) \in C;\ d \in D \tag{8}$$

Constraints (5) make sure that each empty cell is assigned exactly one digit. Then constraints (6) take care of the rule that all digits belonging to a sum must be different, that is, each digit may occur at most once in a sum. According to constraints (7), the digits must add up to the required sum. Constraints (8) limit the decision variables to binary values.

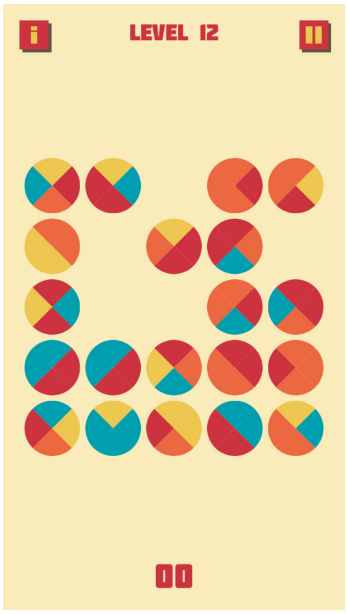
4 Match 22 and ∞ Infinity Loop

In the app *Match 22: Color Puzzle Game*, several disks are given. Each disk is divided into four segments, and each segment is colored in either yellow, orange, red, or blue. The goal is to arrange the disks in a way that the adjacent segments of any two neighboring disks have the same color. To achieve this, the player can turn the disks. Every time he or she tips a disk, it turns left (i.e., counterclockwise) by 90 degrees. The moves (tips) are counted, so one would want to match all colors with the smallest possible number of moves. Figure 3 shows an example level along with the solution.

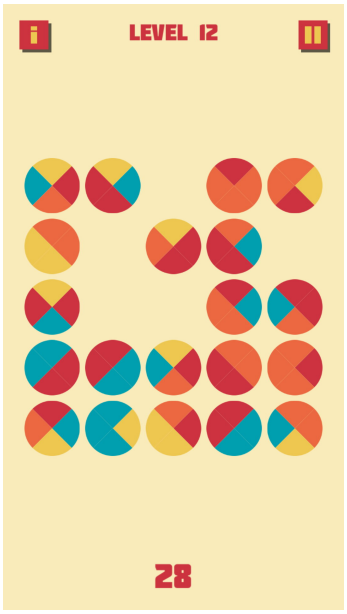
In order to develop a mathematical model, we label the disks using $i = 1,\ldots,n$. For any two horizontally neighboring disks i and k, we introduce an ordered pair (i,k) which means that disk k is the immediate neighbor to the right of disk i. The set of all such pairs is referred to as H. Analogously, V is the set of all pairs of vertical neighbors.

Next, we transform the four colors into numbers; we choose 1 for yellow, 2 for orange, 3 for red, and 4 for blue. The four segments of a disk are labeled 0, 1, 2, 3 according to Figure 3 (c). We can see that for horizontal neighbors $(i,k) \in H$, the color (i.e., number) of segment 3 of disk i must match the color (number) of segment 1 of disk k. Similarly, for vertical neighbors $(i,k) \in V$, the color (number) of segment 2 of disk i has to match the color (number) of segment 0 of disk k.

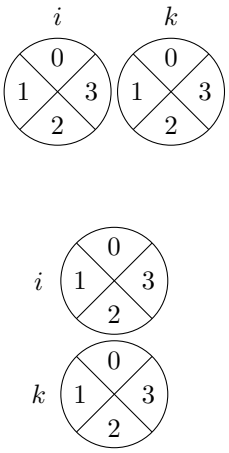
The colors of the disks in the initial setting are captured by parameters d_{ih}, where i is the



(a) Puzzle



(b) Solution



(c) Modeling approach

Figure 3: Smartphone app *Match 22: Color Puzzle Game*

number of the disk and h is the segment index according to Figure 3 (c). Consider the upper left disk in Figure 3 (a), referred to as disk 1. Segment 0 is yellow and segment 3 is red, so we have $d_{10} = 1$ and $d_{13} = 3$. In the model below, turning a disk essentially corresponds to transforming segment index h.

The binary decision variables specify for each disk $i = 1, \ldots, n$ how many times it must be turned left by 90 degrees.

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ left-turns should be applied to disk } i \\ 0 & \text{otherwise} \end{cases}$$

Of course, between 0 and 3 such left turns can be applied to each disk, thus we have $j \in \{0, \ldots, 3\}$. We can now provide a mathematical programming formulation. In contrast to the previous puzzle, this model actually includes an objective. Also note that the model contains a constant for the maximum number of turns. This is because the number of segments of a disk is fixed by the design of the game.

$$\text{Minimize} \sum_{i=1}^{n} \sum_{j=0}^{3} j x_{ij} \tag{9}$$

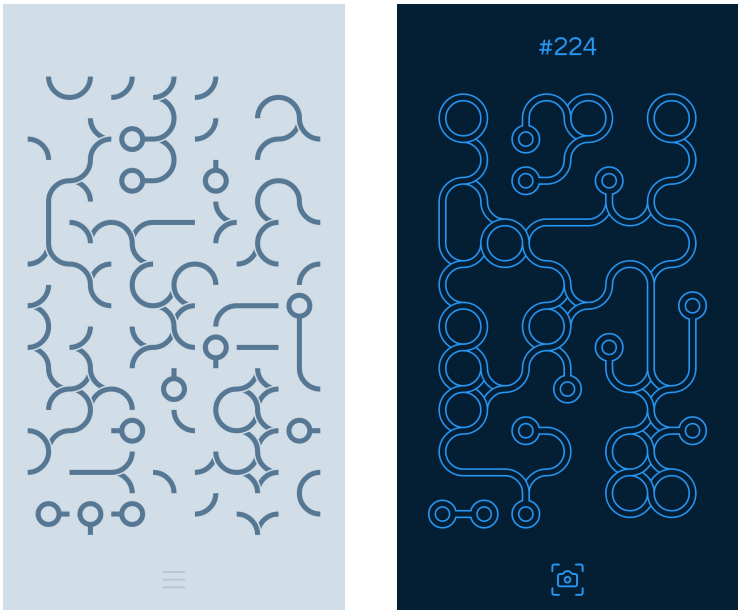
subject to

$$\sum_{j=0}^{3} x_{ij} = 1 \qquad i = 1, \ldots, n \tag{10}$$

$$\sum_{j=0}^{3} d_{i,(3-j) \bmod 4} \cdot x_{ij} = \sum_{j=0}^{3} d_{k,(1-j) \bmod 4} \cdot x_{kj} \qquad (i,k) \in H \tag{11}$$

$$\sum_{j=0}^{3} d_{i,(2-j) \bmod 4} \cdot x_{ij} = \sum_{j=0}^{3} d_{k,(0-j) \bmod 4} \cdot x_{kj} \qquad (i,k) \in V \tag{12}$$

$$x_{ij} \in \{0,1\} \qquad i, = 1, \ldots, n;\ j = 0, \ldots, 3 \tag{13}$$



(a) Puzzle **(b)** Solution

Figure 4: Smartphone app ∞ *Infinity Loop*

Objective (9) minimizes the total number of moves. Constraints (10) make sure that every disk is assigned exactly one number of turns. Subsequently, constraints (11) and (12) observe that horizontally and vertically adjacent segments have the same color, respectively. Finally, constraints (13) restrict the decision variables to be binary.

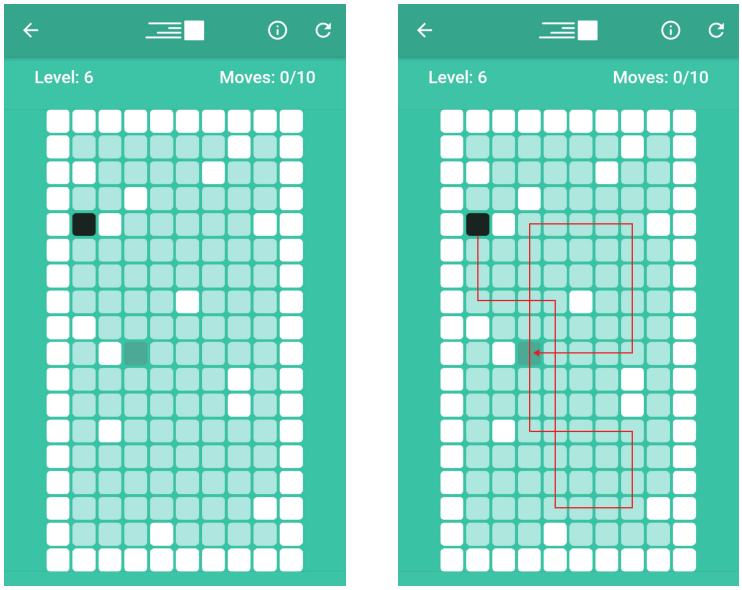
The app ∞ *Infinity Loop* is based on a similar idea. As in *Match 22*, tiles must be turned to solve the puzzle. Here, however, each tile contains up to four connections (to the left, to the right, up, and down), and the tiles must be turned such that an overall pattern without any open connections is achieved. In Figure 4, an example level as well as the related solution are displayed.

We can apply a rather straightforward extension of the mathematical model described above for *Match 22* to solve ∞ *Infinity Loop*. Of course, each disk corresponds to a tile, but instead of colors, we just have to capture in the parameters whether there is a connection in segment h of tile i ($d_{ih} = 1$) or not ($d_{ih} = 0$). We use the same binary decision variables x_{ij} as above to describe the number of turns of each tile. To find a valid solution, we have to take into account an additional requirement that does not appear in *Match 22*. In ∞ *Infinity Loop* there is a restriction that the outer tiles must be turned in a way that no open connection points to the outside of the grid. To capture this, we define sets B_{top}, B_{left}, B_{bottom} and B_{right} containing the border tiles at the top, on the left, at the bottom, and on the right, respectively. Note that, for example, the top left cell will be in both B_{top} and B_{left}. Now we add the following constraints to take this restriction into account, and we obtain (9)–(17) as a mathematical model for ∞ *Infinity Loop*:

$$\sum_{j=0}^{3} d_{i,(0-j) \bmod 4} \cdot x_{ij} = 0 \qquad i \in B_{top} \tag{14}$$

$$\sum_{j=0}^{3} d_{i,(1-j) \bmod 4} \cdot x_{ij} = 0 \qquad i \in B_{left} \tag{15}$$

$$\sum_{j=0}^{3} d_{i,(2-j) \bmod 4} \cdot x_{ij} = 0 \qquad i \in B_{bottom} \tag{16}$$



(a) Puzzle **(b)** Solution

Figure 5: Smartphone app *Slider*

$$\sum_{j=0}^{3} d_{i,(3-j) \bmod 4} \cdot x_{ij} = 0 \qquad i \in B_{right} \tag{17}$$

5 Slider

In the *Slider* app, a rectangle consisting of rows and columns is given. One of the cells is the starting point. It contains a black block that must be moved to another cell, the destination which is marked by a gray cell (see Figure 5); the white cells are obstacles.

The player has to wipe across the smartphone display to move the block. It can be moved either left or right within the row or up or down within the column. It is not possible to move the block by just a few cells. When moved in a direction, the block will always stop at the last cell of the rectangle in the respective row or column, or at the cell immediately before an obstacle. The player solves the puzzle if he or she is able to move the block to the target cell. Beyond that, the goal is to accomplish this with the minimum number of moves.

Let m be the number of rows of the rectangle, and let n be the number of columns (we do not include the white cells on the border, see Figure 5). Then we denote the starting cell by (i_s, j_s) and the destination cell by (i_t, j_t). We can now model the slider puzzle as a standard shortest path problem by defining an appropriate directed graph $G = (V, E)$. The set of vertices or nodes is defined as $V = \{(i, j) \mid i = 1, \ldots, m;\ j = 1, \ldots, n\}$, that is, each cell is a node in the graph. The set E of edges is defined as follows: For each cell (i, j), there is an arc to each cell (k, l) that can be reached from (i, j) in a single move. The edges are then written as (i, j, k, l). Let $S_{ij} = \{(k, l) \mid (i, j, k, l) \in E\}$ represent the set of successors of node (i, j).

If there is no obstacle, an inner cell (i, j) with $1 < i < m$ and $1 < j < n$ has successors $(i, 1)$, (i, n), $(1, j)$ and (m, j). Of course, cells on the border are associated with less than four moves. In case there is an obstacle in cell, say, $(i, j + 3)$, then a move to the right would have to stop before that obstacle, hence the arc representing the move to the right would be $(i, j + 2)$ instead of (i, n). If the obstacle is in $(i, j + 1)$, then there is no possible move to the right and thus no such arc. Finally, there are no edges leading out of the target cell (i_t, j_t), and cells representing obstacles are not associated with any edges.

We need binary decision variables that determine which move should be made and thus which edges in the graph should be selected:

$$x_{ijkl} = \begin{cases} 1 & \text{if the move from cell } (i,j) \text{ to cell } (k,l) \text{ should be made} \\ 0 & \text{otherwise} \end{cases}$$

Now we are ready to give the mathematical programming formulation:

$$\text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{(k,l) \in S_{ij}} x_{ijkl} \tag{18}$$

subject to

$$\sum_{(k,l) \in S_{i_s j_s}} x_{i_s j_s kl} = 1 \tag{19}$$

$$\sum_{\substack{(k,l) \in V \\ (i,j) \in S_{kl}}} x_{klij} = \sum_{(k,l) \in S_{ij}} x_{ijkl} \qquad (i,j) \in V \setminus \{(i_s,j_s),(i_t,j_t)\} \tag{20}$$

$$x_{ijkl} \in \{0,1\} \qquad i,= 1,\ldots,m;\ j = 1,\ldots,n;\ (k,l) \in S_{ij} \tag{21}$$

Objective (18) minimizes the number of moves. Constraints (19) forces the black block to leave the starting cell, and constraints (20) are the usual flow constraints for all nodes except for the source and the destination. The variables are defined to be binary by constraints (21).

6 Flow Free

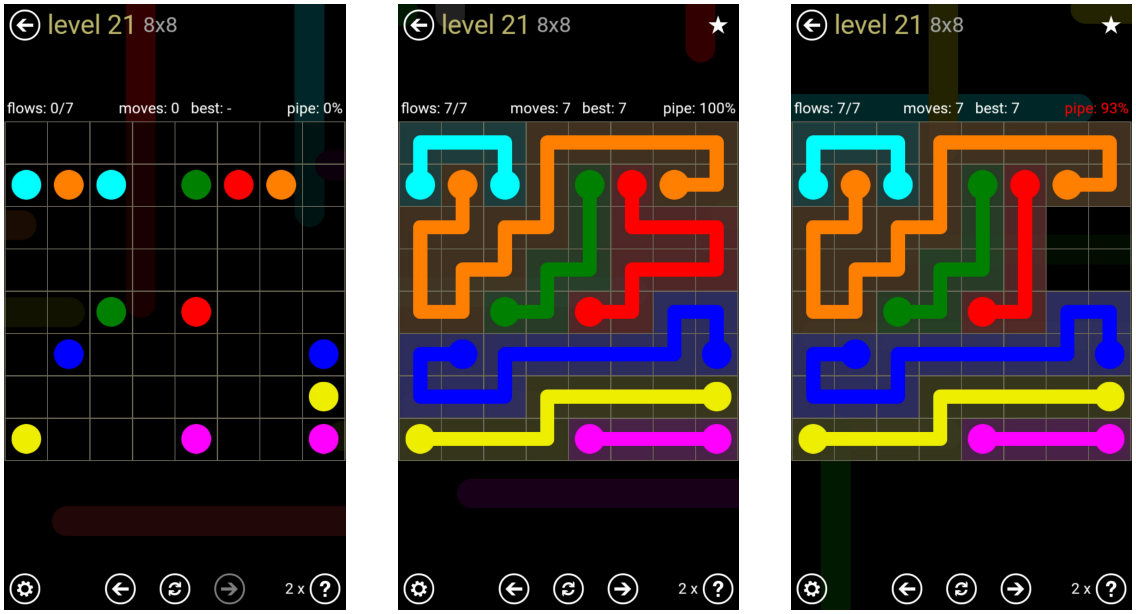
In the app *Flow Free*, a square grid is given. Some cells contain colored dots; there are always two dots of the same color. The player has to connect any two dots of the same color by a continuous line, see Figure 6. Thereby, several rules have to be observed: A line always moves either horizontally or vertically from cell to cell of the grid, diagonal lines are not permitted. All lines must start or end in a cell with a dot, and is not possible that a line enters a cell with a dot and leaves it again. Moreover, lines are not allowed to intersect, that is, each cell of the grid may be associated only with a single line. Each cell must be connected by a line (thus the solution of Figure 6 (c) is invalid). Finally, each line must be connected to two dots, that is, additional lines are not possible. There are also other apps with similar puzzles.

We will model this puzzle as a network flow problem (which nicely matches the name of the app). It can be viewed as a multi-commodity flow problem where each color is represented by a commodity and each pair of dots of the same color corresponds to the source and destination of a commodity. First, we define a directed graph $G = (V, E)$. Each cell of the $n \times n$ grid corresponds to a node which is referred to by (i, j). The set of neighbor nodes of a node (i, j) is given by $N_{ij} = \{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\} \cap V$. The set E of edges is then obtained by defining an edge (i, j, k, l) between any two neighbor nodes $(i, j) \in V$ and $(k, l) \in N_{ij}$.

Next, one dot of each color is defined as source node and the other is defined as destination node. The sets of sources and destination nodes are denoted as S and T, respectively. We also have to take care of the colors which are transformed into values. The color of the first source and destination is 1, that of the second is 2 and so on. These color values are captured by parameters C_{ij} for every source and every destination node $(i, j) \in S \cup T$. We have Q different colors, $Q = |S|$.

Let us now define the decision variables. We consider binary variables that take into account both the logical flow (i.e., to avoid intersecting lines and to make sure that every cell is connected) and the colors of the lines:

$$x_{qijkl} = \begin{cases} 1 & \text{if a line of color } q \text{ should be drawn from cell } (i,j) \text{ to cell } (k,l) \\ 0 & \text{otherwise} \end{cases}$$



(a) Puzzle **(b)** Solution **(c)** Invalid solution

Figure 6: Smartphone app *Flow Free*

Moreover, we have to avoid cycles in the solution. Consider the four empty cells on the right of Figure 6 (c). The flow constraints could be fulfilled by creating a cycle through these four cells, but this would not be allowed by the rules. In order to solve this issue, we introduce variables z_{ij} for all nodes $(i,j) \in V$. The related constraints are explained below. Now we are ready to present the model:

$$\sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} x_{qklij} = 0 \qquad (i,j) \in S \tag{22}$$

$$\sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} x_{qijkl} = 0 \qquad (i,j) \in T \tag{23}$$

$$\sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} x_{qklij} = 1 \qquad (i,j) \in V \setminus S \tag{24}$$

$$\sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} x_{qijkl} = 1 \qquad (i,j) \in V \setminus T \tag{25}$$

$$\sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} q x_{qijkl} = C_{ij} \qquad (i,j) \in S \tag{26}$$

$$\sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} q x_{qklij} = C_{ij} \qquad (i,j) \in T \tag{27}$$

$$\sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} q x_{qklij} = \sum_{q=1}^{Q} \sum_{(k,l)\in N_{ij}} q x_{qijkl} \qquad (i,j) \in V \setminus (S \cup T) \tag{28}$$

$$z_{kl} - z_{ij} + M(1 - \sum_{q=1}^{Q} x_{qijkl}) \geq 1 \qquad (i,j) \in V,\ (k,l) \in N_{ij} \tag{29}$$

$$x_{qijkl} \in \{0,1\} \qquad q = 1,\ldots,Q,\ (i,j) \in V,\ (k,l) \in N_{ij} \tag{30}$$
$$z_{ij} \geq 0 \qquad (i,j) \in V \tag{31}$$

Constraints (22) and (23) make sure that there is no flow into a source and out of a destination node, respectively. Then constraints (24) imply that there is a flow of 1 into each node that is not a source. Analogously, constraints (25) enforce a flow of 1 out of each node that is not a destination node. Note that these two constraints also imply that there is a flow through each node and that flows do not intersect. Next, constraints (26) and (27) indicate that there is a flow out of each source and into each destination, respectively, that corresponds to the related color value. The usual flow constraints are given by (28). They make sure that the color that goes into a node will also go out of it.

Moreover, constraints (29) link the x and the z variables. They state that each node must be associated with a z value that is larger than that of its predecessor. This way, cycles are made impossible, and every flow must be connected to a source and a destination node. Of course, these are the subtour elimination constraints proposed by Miller et al. (1960) for the traveling salesman problem (TSP). M is a large number; note that we can set $M = n^2$. The last two constraints (30) and (31) define the decision variables.

It might be interesting to look at another possible modeling approach for the *Flow Free* app. In the model above, we have captured both the logical flow in the color flow by binary decision variables. The idea is to split this: Now the logical flow (i.e., the decision whether there is a line from one cell to another or not) is taken care of by binary decision variables related to the edges:

$$x_{ijkl} = \begin{cases} 1 & \text{if a line should be drawn from cell } (i,j) \text{ to cell } (k,l) \\ 0 & \text{otherwise} \end{cases}$$

The color of the flow (i.e., the color of the line) from a cell (i,j) to a cell (k,l) is now captured by variables y_{ijkl} which don't have to be binary or integer. That is, while variables y_{ijkl} take the color of the flow into account, variables x_{ijkl} are needed to make sure that each cell is connected and that flows (lines) do not intersect. Finally, we need again decision variables z_{ij} for all nodes $(i,j) \in V$ that help to avoid cycles in the solution. The resulting model can be stated as follows:

$$\sum_{(k,l)\in N_{ij}} x_{klij} = 0 \qquad (i,j) \in S \tag{32}$$
$$\sum_{(k,l)\in N_{ij}} x_{ijkl} = 0 \qquad (i,j) \in T \tag{33}$$
$$\sum_{(k,l)\in N_{ij}} x_{klij} = 1 \qquad (i,j) \in V \setminus S \tag{34}$$
$$\sum_{(k,l)\in N_{ij}} x_{ijkl} = 1 \qquad (i,j) \in V \setminus T \tag{35}$$
$$\sum_{(k,l)\in N_{ij}} y_{ijkl} = C_{ij} \qquad (i,j) \in S \tag{36}$$
$$\sum_{(k,l)\in N_{ij}} y_{klij} = C_{ij} \qquad (i,j) \in T \tag{37}$$
$$\sum_{(k,l)\in N_{ij}} y_{klij} = \sum_{(k,l)\in N_{ij}} y_{ijkl} \qquad (i,j) \in V \setminus (S \cup T) \tag{38}$$
$$y_{ijkl} \geq x_{ijkl} \qquad (i,j) \in V,\ (k,l) \in N_{ij} \tag{39}$$
$$y_{ijkl} \leq M_1 x_{ijkl} \qquad (i,j) \in V,\ (k,l) \in N_{ij} \tag{40}$$
$$z_{kl} - z_{ij} + M_2(1 - x_{ijkl}) \geq 1 \qquad (i,j) \in V,\ (k,l) \in N_{ij} \tag{41}$$
$$x_{ijkl} \in \{0,1\} \qquad (i,j) \in V,\ (k,l) \in N_{ij} \tag{42}$$

$$y_{ijkl} \geq 0 \qquad (i,j) \in V,\ (k,l) \in N_{ij} \tag{43}$$

$$z_{ij} \geq 0 \qquad (i,j) \in V \tag{44}$$

The first four contraints (32)–(35) represent the logical flow. They are analogous to contraints (22)–(25). The colors are now observed by the y variables. Constraints (36), (37), and (38) take care of the colors leaving the sources, entering the destinations, and flowing through the remaining nodes, respectively.

Next, constraints (39) and (40) link the x and the y variables. They say that there is a flow on an edge according to the associated y variable if and only if there is a flow according to the associated x variable. M_1 is a large number; $M_1 = Q$ is a natural choice if the colors of the flows are transformed into values $1, 2, \ldots, Q$. Then constraints (41) link the x and the z variables. Similarly to constraints (29) of the first model, they make sure that no cycles occur (like above, M_2 is a large number which can be set to $M_2 = n^2$). Finally, the remaining three constraints (42)–(44) define the decision variables.

The two models presented for the *Flow Free* app can be used to discuss the impact of modeling techniques. In the first approach, both the logical flow and the colors are captured by binary variables. In contrast, the second approach employs binary variables only for the logical flow while continuous variables are used for the colors. Obviously, the first model contains $Q \cdot n^4$ binary variables whereas the second one contains only n^4.

Often models with fewer binary variables lead to shorter solution times when solvers are applied. This is also the case here: We solved the example of Figure 6 (a) using the solver of the GNU Linear Programming Kit (GLPK, see Makhorin (2010)) on a computer with an Intel i5 CPU running at 3.0 GHz. The solution time of the first model (22)–(31) was 324.7 seconds while the solution time of the second model (32)–(44) was only 0.4 seconds.

7 Conclusions

In this paper, we have presented six smartphone apps that provide different logic puzzles. We developed integer (and mixed integer) programming models that can be applied to determine solutions for these puzzles.

The puzzle apps and models are used in the introductory OR/MS lectures in BSc programs in management. Considering this, a particular focus was on apps that allow for models with a difficulty level between easy and moderate. Some puzzles (usually *Thermometer Puzzles* and *Kakuro*) are given to the students as additional modeling exercises they can do at home if they like. In evaluations of the lectures, the exercises generally get positive feedback. While this applies to the exercises in general and not only to the puzzle exercises in particular, our impression is that most students actually do the puzzle exercises (and many find them interesting and motivating). The other puzzles are used for classroom discussions whenever there is enough time. Questions like the following are discussed: What would the variables/the underlying graph/the constraints look like? What are the similarities to other models, what ideas from other models can be used here again? We find these kinds of discussions helpful to teach IP modeling because they support a deeper understanding (which is often not achieved if students try to learn given models by heart). In the discussions, the puzzles are also used to point out that mathematical models and solvers are actually very general tools that can solve a broad variety of problems. Thus, puzzle-based exercises and discussions have become a regular part of our introductory OR/MS lectures.

Models and example data implemented in MathProg (Makhorin, 2010) can be found in the appendix. It may be interesting to add that all instances provided in the appendix could be solved within less than 0.5 seconds except for the *Kakuro* instance of Figure 2 which required 4.5 seconds and the the first model for the *Flow Free* instance of Figure 6 which required Figure 324.7 seconds (all tests were carried out on a computer with an Intel i5 CPU running at 3.0 GHz).

While we currently use the apps only for teaching mathematical modeling, it should be mentionened that some of them might also be used as examples when teaching other OR techniques such as branch-and-bound or constraint programming.

Acknowledgements. The author thanks the developers of the apps for their kind permissions to reproduce the screenshots. *Thermometer Puzzles*: Francois Guibert (Frozax Games, France); *Kakuro (Cross Sums)*: Pink Pointer Games (Brazil); *Match 22*: Abhinav Bhadoria (Zero Logic Games, India); ∞ *Infinity Loop*: Muhammad Satar (Webavenue, Portugal) and Jonas Lekevicius and Balys Valentukevicius (Lithuania); *Slider*: Alex Gwyn (USA); *Flow Free*: Sharon Newman (Big Duck Games LLC, USA).

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A MathProg File for Thermometer Puzzles

The code given below corresponds to the model introduced in Section 2. The data part captures the example displayed in Figure 1.

```
param n integer;
param R{1..n} integer;
param C{1..n} integer;
set   D within {1..n} cross {1..n} cross {1..n} cross {1..n};

var x{1..n,1..n} binary;

maximize F:  0;

s.t. RowCount     {i in 1..n}:       sum{j in 1..n}  x[i,j]  = R[i];
s.t. ColCount     {j in 1..n}:       sum{i in 1..n}  x[i,j]  = C[j];
s.t. Dependencies {(i,j,k,l) in D}:                  x[i,j] >= x[k,l];

solve;

printf"\nSolution:\n\n";
for {i in 1..n}
{
  printf{j in 1..n} x[i,j];
  printf"\n";
}
printf"\n";

data;

param n := 8;

param :  R  C :=
1        6  4
2        5  6
3        4  4
4        3  6
5        6  4
6        3  6
7        7  7
8        5  2 ;

set D := (1,5,1,6)  (1,6,1,7)
         (4,3,4,2)
         (5,2,5,3)  (5,3,5,4)  (5,4,5,5)  (5,5,5,6)  (5,6,5,7)
         (6,2,6,3)  (6,3,6,4)  (6,4,6,5)  (6,5,6,6)  (6,6,6,7)
         (7,3,7,2)  (7,7,7,6)  (7,6,7,5)  (7,5,7,4)
         (8,1,8,2)  (8,2,8,3)  (8,3,8,4)  (8,7,8,6)  (8,6,8,5)
         (4,1,3,1)  (3,1,2,1)  (2,1,1,1)  (7,1,6,1)  (6,1,5,1)
         (1,2,2,2)  (2,2,3,2)
         (1,3,2,3)  (2,3,3,3)
         (1,4,2,4)  (2,4,3,4)  (3,4,4,4)
         (4,5,3,5)  (3,5,2,5)
         (2,6,3,6)  (3,6,4,6)
         (4,7,3,7)  (3,7,2,7)
         (8,8,7,8)  (7,8,6,8)  (6,8,5,8)  (5,8,4,8)  (4,8,3,8)  (3,8,2,8)  (2,8,1,8) ;

end;
```

B MathProg File for Kakuro (Cross Sums)

In what follows, we give the MathProg code resulting from the model presented in Section 3. The data part corresponds to the example given in Figure 2.

```
param n integer;
param K integer;
param r{1..K} integer;
set   R{1..K} within {1..n} cross {1..n};
set   C within {1..n} cross {1..n};

var x{(i,j) in C, 1..9} binary;

maximize F:  0;

s.t. Digit     {(i,j) in C}:            sum{d in 1..9}      x[i,j,d]   =  1;
s.t. Different {k in 1..K, d in 1..9}:  sum{(i,j) in R[k]}  x[i,j,d]  <=  1;
s.t. Sum       {k in 1..K}:             sum{(i,j) in R[k], d in 1..9}  d * x[i,j,d]  =  r[k];

solve;

printf"\nSolution:\n\n";
for {i in 1..n}
{
  for {j in 1..n} printf if (i,j) in C then sum{d in 1..9}  d * x[i,j,d] else " ";
  printf"\n";
}
printf"\n";

data;

param n :=  8;
param K := 30;

param r :=
1  11
2  14
3  34
4  23
5   8
6  23
7  11
8   6
9  14
10  3
11 10
12 36
13  8
14 30
15 12
16 10
17  6
18 18
19 31
20 13
21 12
22  7
23  8
24 17
25 14
26 26
27 32
28  7
29  8
30 10 ;

set R[ 1] := (1,1) (1,2) (1,3) (1,4);
set R[ 2] := (1,6) (1,7);
set R[ 3] := (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (2,7);
set R[ 4] := (3,3) (3,4) (3,5);
set R[ 5] := (3,7) (3,8);
```

```
set R[ 6] := (4,1) (4,2) (4,3);
set R[ 7] := (4,6) (4,7) (4,8);
set R[ 8] := (5,1) (5,2) (5,3);
set R[ 9] := (5,6) (5,7) (5,8);
set R[10] := (6,1) (6,2);
set R[11] := (6,4) (6,5) (6,6);
set R[12] := (7,2) (7,3) (7,4) (7,5) (7,6) (7,7) (7,8);
set R[13] := (8,2) (8,3);
set R[14] := (8,5) (8,6) (8,7) (8,8);
set R[15] := (1,1) (2,1);
set R[16] := (4,1) (5,1) (6,1);
set R[17] := (1,2) (2,2);
set R[18] := (4,2) (5,2) (6,2) (7,2) (8,2);
set R[19] := (1,3) (2,3) (3,3) (4,3) (5,3);
set R[20] := (7,3) (8,3);
set R[21] := (1,4) (2,4) (3,4);
set R[22] := (6,4) (7,4);
set R[23] := (2,5) (3,5);
set R[24] := (6,5) (7,5) (8,5);
set R[25] := (1,6) (2,6);
set R[26] := (4,6) (5,6) (6,6) (7,6) (8,6);
set R[27] := (1,7) (2,7) (3,7) (4,7) (5,7);
set R[28] := (7,7) (8,7);
set R[29] := (3,8) (4,8) (5,8);
set R[30] := (7,8) (8,8);

set C : 1 2 3 4 5 6 7 8 :=
1       + + + + - + + -
2       + + + + + + + -
3       - - + + + - + +
4       + + + - - + + +
5       + + + - - + + +
6       + + - + + + - -
7       - + + + + + + +
8       - + + - + + + + ;

end;
```

This MathProg Code will print the following solution (compare to Figure 2 b):

```
Solution:

3152 86
9571264
  896 53
689  281
312  194
12 136
 4865913
 35 9867
```

C MathProg File for Match 22

The following code corresponds to the model of Section 4. The data part corresponds to the example given in Figure 3.

```
param n integer;
set   H within {1..n} cross {1..n};
set   V within {1..n} cross {1..n};
param d{1..n, 0..3};

var   x{1..n,0..3} binary;
```

```
minimize F:  sum{i in 1..n, j in 0..3}  j * x[i,j];

s.t. OneOrientation{i in 1..n}:
     sum{j in 0..3}  x[i,j]  =  1;
s.t. Horizontal    {(i,k) in H}:
     sum{j in 0..3}  d[i,(3-j) mod 4] * x[i,j]  =  sum{j in 0..3}  d[k,(1-j) mod 4] * x[k,j];
s.t. Vertical      {(i,k) in V}:
     sum{j in 0..3}  d[i,(2-j) mod 4] * x[i,j]  =  sum{j in 0..3}  d[k,(0-j) mod 4] * x[k,j];

solve;

printf"\Solution:\n\n";
for {i in 1..n}
  printf"Disk \%2d: \%d left turn(s)\n", i, sum{j in 0..3} j*x[i,j];
printf"\n\%d left turns in total\n\n", sum{i in 1..n, j in 0..3} j*x[i,j];

data;

param n := 20;

set H := (1,2)  (3,4)  (6,7)  (9,10)  (11,12)  (12,13)  (13,14)
         (14,15)  (16,17)  (17,18)  (18,19)  (19,20) ;
set V := (1,5)  (3,7)  (5,8)  (7,9)  (8,11)  (9,14)  (10,15)
         (11,16)  (12,17)  (13,18)  (14,19)  (15,20) ;

param d :  0 1 2 3 :=
1          1 4 2 3
2          1 3 3 4
3          2 2 2 3
4          2 2 3 1
5          2 1 1 2
6          1 2 3 3
7          3 3 4 2
8          3 1 3 4
9          2 2 4 3
10         3 4 2 3
11         4 4 3 3
12         4 4 3 3
13         3 1 4 2
14         3 2 2 3
15         2 3 2 2
16         4 3 2 1
17         1 4 4 4
18         1 3 2 1
19         4 3 3 4
20         1 2 2 4 ;

end;
```

This MathProg file leads to the output below which corresponds to the solution shown in Figure 3 (b):

```
Solution:

Disk  1: 0 left turn(s)
Disk  2: 0 left turn(s)
Disk  3: 1 left turn(s)
Disk  4: 0 left turn(s)
Disk  5: 0 left turn(s)
Disk  6: 0 left turn(s)
Disk  7: 1 left turn(s)
Disk  8: 3 left turn(s)
Disk  9: 1 left turn(s)
Disk 10: 0 left turn(s)
Disk 11: 0 left turn(s)
```

```
Disk 12: 2 left turn(s)
Disk 13: 3 left turn(s)
Disk 14: 2 left turn(s)
Disk 15: 2 left turn(s)
Disk 16: 3 left turn(s)
Disk 17: 3 left turn(s)
Disk 18: 2 left turn(s)
Disk 19: 3 left turn(s)
Disk 20: 2 left turn(s)

28 left turns in total
```

D MathProg File for Slider

The following MathProg code contains the model of Section 5. The data part corresponds to a very small example shown in Figure 7 (note that this is not taken from the Slider app—the orginal levels of the app would have led to much larger data parts in the MathProg file). Also note that the border cells are not part of the data.

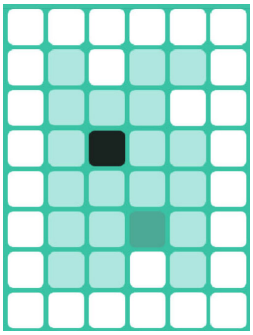

Figure 7: Small example based on *Slider*

```
param m   integer;
param n   integer;
param i_s integer;
param j_s integer;
param i_t integer;
param j_t integer;
set   ST  within {1..m} cross {1..n} := {(i_s,j_s), (i_t,j_t)};
set   S{i in 1..m, j in 1..n} within {1..m} cross {1..n};

var x{i in 1..m, j in 1..n, k in 1..m, l in 1..n : (k,l) in S[i,j]}  binary;

minimize F: sum{i in 1..m, j in 1..n}  sum{k in 1..m, l in 1..n : (k,l) in S[i,j]}  x[i,j,k,l];

s.t. Flow{i in 1..m, j in 1..n : (i,j) not in ST}:
        sum{k in 1..m, l in 1..n : (i,j) in S[k,l]}  x[k,l,i,j]
     =  sum{k in 1..m, l in 1..n : (k,l) in S[i,j]}  x[i,j,k,l];
s.t. Source:
     sum{k in 1..m, l in 1..n : (k,l) in S[i_s,j_s]}  x[i_s,j_s,k,l]  =  1;

solve;

printf"\nSolution (moves may be given in arbitrary order):\n\n";
for {i in 1..m}
  for {j in 1..n}
    printf {(k,l) in S[i,j] : x[i,j,k,l] = 1} : "(\%d,\%d) --> (\%d,\%d)\n", i, j, k, l;
printf"\n";
```

```
data;

param m := 6;
param n := 4;

param i_s := 3;
param j_s := 2;
param i_t := 5;
param j_t := 3;

set S[1,1]  :=  (6,1) ;
set S[2,1]  :=  (1,1)  (6,1)  (2,3) ;
set S[3,1]  :=  (1,1)  (6,1)  (3,4) ;
set S[4,1]  :=  (1,1)  (6,1)  (4,4) ;
set S[5,1]  :=  (1,1)  (6,1)  (5,4) ;
set S[6,1]  :=  (1,1)  (6,1)  (6,2) ;

set S[1,2]  :=  ;
set S[2,2]  :=  (6,2)  (2,1)  (2,3) ;
set S[3,2]  :=  (2,2)  (6,2)  (3,1)  (3,4) ;
set S[4,2]  :=  (2,2)  (6,2)  (4,1)  (4,4) ;
set S[5,2]  :=  (2,2)  (6,2)  (5,1)  (5,4) ;
set S[6,2]  :=  (2,2)  (6,1) ;

set S[1,3]  :=  (5,3)  (1,4) ;
set S[2,3]  :=  (1,3)  (5,3)  (2,1) ;
set S[3,3]  :=  (1,3)  (5,3)  (3,1)  (3,4) ;
set S[4,3]  :=  (1,3)  (5,3)  (4,1)  (4,4) ;
set S[5,3]  :=  ;
set S[6,3]  :=  ;

set S[1,4]  :=  (1,3) ;
set S[2,4]  :=  ;
set S[3,4]  :=  (6,4)  (3,1) ;
set S[4,4]  :=  (3,4)  (6,4)  (4,1) ;
set S[5,4]  :=  (3,4)  (6,4)  (5,1) ;
set S[6,4]  :=  (3,4) ;

end;
```

This MathProg file produces the following output:

```
Solution (moves may be given in arbitrary order):

(2,2) --> (2,3)
(2,3) --> (5,3)
(3,2) --> (2,2)
```

E MathProg File for Flow Free

The MathProg code given below contains the first model of Section 6. The data section reflects the example of Figure 6 (a).

```
param n integer;
param Q integer;
set   V := {1..n} cross {1..n};
set   N{(i,j) in V} within V := {(i-1,j), (i+1,j), (i,j-1), (i,j+1)} inter {1..n} cross {1..n};
set   S within {1..n} cross {1..n};
set   T within {1..n} cross {1..n};
param C{1..n, 1..n} integer;

var x{q in 1..Q, (i,j) in V, (k,l) in N[i,j]} binary;
```

```
var z{(i,j) in V} >= 0;

minimize F:  0;

s.t. FlowInSources {(i,j) in S}:         sum{q in 1..Q}  sum{(k,l) in N[i,j]}  x[q,k,l,i,j]      =  0;
s.t. FlowOutSinks  {(i,j) in T}:         sum{q in 1..Q}  sum{(k,l) in N[i,j]}  x[q,i,j,k,l]      =  0;
s.t. FlowInOther   {(i,j) in V diff S}:  sum{q in 1..Q}  sum{(k,l) in N[i,j]}  x[q,k,l,i,j]      =  1;
s.t. FlowOutOther  {(i,j) in V diff T}:  sum{q in 1..Q}  sum{(k,l) in N[i,j]}  x[q,i,j,k,l]      =  1;
s.t. FlowOutSources{(i,j) in S}:         sum{q in 1..Q}  sum{(k,l) in N[i,j]}  q * x[q,i,j,k,l]  =  C[i,j];
s.t. FlowInSinks   {(i,j) in T}:         sum{q in 1..Q}  sum{(k,l) in N[i,j]}  q * x[q,k,l,i,j]  =  C[i,j];
s.t. Flow          {(i,j) in V diff (S union T)}:  sum{q in 1..Q}  sum{(k,l) in N[i,j]}  q * x[q,k,l,i,j]
                                                 =  sum{q in 1..Q}  sum{(k,l) in N[i,j]}  q * x[q,i,j,k,l];
s.t. xz_Link       {(i,j) in V, (k,l) in N[i,j]}:  z[k,l] - z[i,j] + n * n * (1 - (sum{q in 1..Q} x[q,i,j,k,l]))
                                                >=  1;

solve;

printf"\nSolution:\n\n";
for {i in 1..n}
{
  for {j in 1..n}
    printf "\%d", if (i,j) in S then sum{q in 1..Q}  sum{(k,l) in N[i,j]}
    q*x[q,i,j,k,l] else sum{q in 1..Q}  sum{(k,l) in N[i,j]}  q*x[q,k,l,i,j];
  printf"\n";
}
printf"\n";

data;

param n := 8;
param Q := 7;
set   S := (2,1) (2,2) (2,5) (2,6) (6,2) (8,1) (8,5);
set   T := (2,3) (2,7) (5,3) (5,5) (6,8) (7,8) (8,8);

param C : 1  2  3  4  5  6  7  8  :=
1         0  0  0  0  0  0  0  0
2         1  2  1  0  3  4  2  0
3         0  0  0  0  0  0  0  0
4         0  0  0  0  0  0  0  0
5         0  0  3  0  4  0  0  0
6         0  5  0  0  0  0  0  5
7         0  0  0  0  0  0  0  6
8         6  0  0  0  7  0  0  7 ;

end;
```

The MathProg code for the second model for the *Flow Free* app is as follows:

```
param n integer;
set   V := {1..n} cross {1..n};
set   N{(i,j) in V} within V := {(i-1,j), (i+1,j), (i,j-1), (i,j+1)} inter {1..n} cross {1..n};
set   S within {1..n} cross {1..n};
set   T within {1..n} cross {1..n};
param C{1..n, 1..n} integer;

var x{(i,j) in V, (k,l) in N[i,j]} binary;
var y{(i,j) in V, (k,l) in N[i,j]} >= 0;
var z{(i,j) in V} >= 0;

minimize F:  0;

s.t. x_FlowInSources {(i,j) in S}:                 sum{(k,l) in N[i,j]}  x[k,l,i,j]  =  0;
s.t. x_FlowOutSinks  {(i,j) in T}:                 sum{(k,l) in N[i,j]}  x[i,j,k,l]  =  0;
s.t. x_FlowInOther   {(i,j) in V diff S}:          sum{(k,l) in N[i,j]}  x[k,l,i,j]  =  1;
s.t. x_FlowOutOther  {(i,j) in V diff T}:          sum{(k,l) in N[i,j]}  x[i,j,k,l]  =  1;
```

```
s.t. y_FlowOutSources{(i,j) in S}:                  sum{(k,l) in N[i,j]}  y[i,j,k,l]  =  C[i,j];
s.t. y_FlowInSinks   {(i,j) in T}:                  sum{(k,l) in N[i,j]}  y[k,l,i,j]  =  C[i,j];
s.t. y_Flow          {(i,j) in V diff (S union T)}: sum{(k,l) in N[i,j]}  y[k,l,i,j]
                                                  =  sum{(k,l) in N[i,j]}  y[i,j,k,l];
s.t. xy_Link1        {(i,j) in V, (k,l) in N[i,j]}: y[i,j,k,l]  >=           x[i,j,k,l];
s.t. xy_Link2        {(i,j) in V, (k,l) in N[i,j]}: y[i,j,k,l]  <=  card(S) * x[i,j,k,l];
s.t. xz_Link         {(i,j) in V, (k,l) in N[i,j]}: z[k,l] - z[i,j] + n*n*(1-x[i,j,k,l])  >=  1;

solve;

printf"\nSolution:\n\n";
for {i in 1..n}
{
  for {j in 1..n}
    printf "\%d", if (i,j) in S then sum{(k,l) in N[i,j]}  y[i,j,k,l] else sum{(k,l) in N[i,j]}  y[k,l,i,j];
  printf"\n";
}
printf"\n";

data;

param n := 8;
set   S := (2,1) (2,2) (2,5) (2,6) (6,2) (8,1) (8,5);
set   T := (2,3) (2,7) (5,3) (5,5) (6,8) (7,8) (8,8);

param C : 1  2  3  4  5  6  7  8  :=
1         0  0  0  0  0  0  0  0
2         1  2  1  0  3  4  2  0
3         0  0  0  0  0  0  0  0
4         0  0  0  0  0  0  0  0
5         0  0  3  0  4  0  0  0
6         0  5  0  0  0  0  0  5
7         0  0  0  0  0  0  0  6
8         6  0  0  0  7  0  0  7 ;

end;
```

This will produce the following output which corresponds to the solution of Figure 6 (b):

```
Solution:

11122222
12123422
22223444
22233444
22334455
55555555
55566666
66667777
```